Pertanika J. Sci. & Technol. 26 (2): 841 - 852 (2018)

PERTANIKA JOURNALS

SCIENCE & TECHNOLOGY

Journal homepage: http://www.pertanika.upm.edu.my/

Application of Almost Increasing Sequence for Absolute Riesz $|\bar{N},p_n^{\alpha};\delta;\gamma|_k$ Summable Factor

Sonker, Smita and Munjal, Alka*

Department of Mathematics, National Institute of Technology Kurukshetra, Kurukshetra-136119, Haryana, India

ABSTRACT

A moderated theorem was established in this study to derive an infinite series as an absolute Riesz $|\bar{N},p_n^{\alpha};\delta;\gamma|_k$ summable by applying an almost increasing sequence. Suitable conditions were applied for obtaining some well-known applications of the presented theorem. It was observed from the investigation that absolute summability could be used for Bounded Input Bounded Output stability and to bind the output of the signals as absolute summable, it was a necessary and sufficient condition for BIBO stability.

Keywords: Almost increasing sequence, absolute summability, infinite series, $|\bar{N},p_n^{\alpha};\delta;\gamma|_k$ summability

INTRODUCTION

Summability is a field that studies the non-convergent series/integrals and assigns a value (number) to it. In mathematical analysis, the summability method is an alternative formulation of convergence of a series that is divergent in the conventional sense. Throughout the 19th century, many mathematicians studied various sets of divergent series and defined numerous summability methods such as Abel summability, Cesàro summability, Euler summability, Hausdorff summability, Nörlund summability, Riesz summability etc. Riesz summability was introduced by Marcel Riesz in 1911 as an improvement to the Cesàro mean. It is a generalisation of Cesàro (C, 1) summability, which can be obtained using $p_n = 1$. The present paper is devoted to the study of the Riesz summability method and a theorem that has been established for the absolute Riesz summable factor of an infinite series.

With the help of a minimal set of sufficient conditions for a soft computing model, error can be minimised by using summability methods so that the output data are filtered according to the user's interest. The output data are given by u_n, which is a sequence-to-sequence transformation. For such computing,

Article history:
Received: 12 December 2017
Accepted: 28 March 2018

E-mail addresses:
smita.sonker@gmail.com (Sonker, Smita)
alkamunjal8@gmail.com (Munjal, Alka)
*Corresponding Author

ISSN: 0128-7680 © 2018 Universiti Putra Malaysia Press.

$\{s_n\}$ should be the sequence of partial sums of an infinite series $\sum_{n=0}^{\infty} a_n$ and the n^{th} sequence-to-sequence transformation (mean) of $\{s_n\}$ be given as u_n i.e.

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k. \tag{1}$$

However, before discussing $|\overline{N}, p_n^{\alpha}; \delta; \gamma|_k$ summability, it would be meaningful to consider some well-known basic summabilities that can be helpful for understanding the $|\overline{N}, p_n^{\alpha}; \delta; \gamma|_k$ summability.

Definition 1

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with a sequence of partial sums $\{s_n\}$ and said to be absolutely summable, if

$$\lim_{n\to\infty} u_n = s, \tag{2}$$

and

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty, \tag{3}$$

where, u_n represents the n^{th} sequence-to-sequence transformation (mean) of $\{s_n\}$.

Definition 2 (Flett, 1957)

If t_n represents the $n^{th}(C,1)$ means of the sequence (na_n), then the series $\sum_{n=0}^{\infty} a_n$ is said to be $|C,1|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{4}$$

Definition 3

Let (p_n) be a sequence with $p_0 > 0$ and $p_n \geq 0$ for $n > 0$

$$P_n = \sum_{v=0}^{n} p_v \to \infty. \tag{5}$$

For $\alpha > -1$, define

$$\varepsilon_0^{\alpha} = 1, \varepsilon_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!}, (n = 1,2,3,...), \tag{6}$$

$$p_n^{\alpha} = \sum_{v=0}^{n} \varepsilon_{n-v}^{\alpha-1} p_v, \tag{7}$$

$$P_n^\alpha = \sum_{v=0}^{n} p_v^\alpha \to \infty,\ n \to \infty, \tag{8}$$

and $P_{-n}^\alpha = p_{-n}^\alpha = 0,\ n \geq 1,$

then, the sequence-to-sequence transformation t_n defines the $(\overline{N}, p_n^\alpha)$ mean of series $\sum a_n$ and is given by

$$t_n = \frac{1}{P_n^\alpha} \sum_{k=0}^{n} p_k^\alpha s_k, P_n^\alpha \neq 0, n \in N \tag{9}$$

and $lim_{n\to\infty} t_n = s,$ and the series is said to be $(\overline{N}, p_n^\alpha)$ summable, generated by the sequence of coefficients $\{p_n^\alpha\}$.

Further, if sequence $\{t_n\}$ is of bounded variation with index $k \geq 1$ i.e.

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{10}$$

then the series $\sum a_n$ is said to be absolutely $(R, p_n^\alpha)_k$ -summable with index k or $|\overline{N}, p_n^\alpha|_k$ -summable to s.

Definition 4

The series is said to be $|\overline{N}, p_n^\alpha; \delta|_k$ summable, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\delta k+k-1} |\Delta t_{n-1}|^k < \infty, \tag{11}$$

and summable $|\overline{N}, p_n^\alpha; \delta; \gamma|_k$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)} |\Delta t_{n-1}|^k < \infty, \tag{12}$$

where, $k \geq 1, \delta \geq 0$ and γ is a real number and

$$\Delta t_n = -\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{v=1}^{n} P_{v-1}^\alpha a_v, n \geq 1. \tag{13}$$

Bor and Seyhan (1999) determined the set of sufficient conditions for an infinite series to be absolutely Riesz $|\overline{N}, p_n^\alpha; \delta|_k$ summable using an almost increasing sequence. Bor and Özarslan (2002) redesigned the problem of Mazhar (1997) under weaker conditions using a quasi β -power increasing sequence, and Bor (2014) generalised the theorem, dealing with a general class of power increasing sequences and absolute Riesz summability factors of an infinite series.

Özgen (2013) used the concept of positive non-decreasing sequence for an infinite series to be absolutely $|A|_k (k \geq 1)$ summable. Sonker and Munjal (2016a) determined the generalised absolute Cesáro summable factor and obtained the minimal sufficient conditions for the

boundedness of an infinite series (2016b). Sonker and Munjal (2017a, 2017b, 2017c, 2017d), and Sonker, Krasniqi and Munjal (2017) also worked out various interesting theorems on the absolute summable factor. Bor (2015) applied absolute Cesáro and Nörlund summability and established two theorems using more general conditions for the infinite series.

KNOWN RESULTS

Using $|\overline{N}, p_n; \delta|_k$ summability, Bor and Seyhan (1999) proved the following theorem with a minimal set of sufficient conditions for an infinite series to be absolutely Riesz summable.

Theorem 2.1 (Bor and Seyhan, 1999)

Let p_n be a sequence of positive numbers such that

$$P_n = O(np_n) \ as \ n \to \infty. \tag{14}$$

Let (X_n) be an almost increasing sequence and assuming that there exists sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{15}$$

$$\beta_n \to 0 \ as \ n \to \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n \leq \infty, \tag{17}$$

$$|\lambda_n| X_n = O(1) \ as \ n \to \infty, \tag{18}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\}, \tag{19}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \ as \ m \to \infty, \tag{20}$$

where, $t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v$, (21)

then, the series $\sum a_n\lambda_n$ is $|\overline{N}, p_n; \delta|_k$ summable for $k \geq 1$ and $0 \leq \delta \leq 1/k$.

MAIN RESULTS

A sequence is of bounded variation i.e. $(\lambda_n) \in BV$, if

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = |\lambda_n - \lambda_{n+1}| < \infty.$$

A positive sequence (g_n) is an almost increasing sequence if $\exists$ is a positive increasing sequence (h_n) and there are two positive constants, *M* and *N s.t.*

$$Mh_n \leq g_n \leq Nh_n.$$

Theorem 3.1

Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) be such that conditions $(15)-(18)$ of theorem 2.1 are satisfied. If the following conditions also satisfy

$$\sum_{n=v+1}^{\infty} \frac{1}{P_{n-1}^{\alpha}} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k} = O\left\{\frac{1}{P_v^{\alpha}} \left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}\right\}, \tag{22}$$

$$\sum_{n=1}^{m} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k} |t_n|^k = O(X_m), \tag{23}$$

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1), \tag{24}$$

and

$$\sum_{n=1}^{m} \frac{1}{n} \left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)} |t_n|^k = O(X_m) \text{ as } m \to \infty, \tag{25}$$

then, the series $\sum a_n\lambda_n$ is $|\overline{N}, p_n^{\alpha}; \delta; \gamma|_k$ summable for $k \geq 1$, $0 \leq \delta \leq 1/k$ and γ is a real number.

Proof of the Theorem 3.1

Let T_n denote the $(\overline{N}, p_n^{\alpha})$ mean of the series $\sum a_n\lambda_n$. Then, by definition and by changing the order of summation, we have

$$T_n = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} p_v^{\alpha} \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n^{\alpha}} \sum_{v=0}^{n} (P_n^{\alpha} - P_{v-1}^{\alpha}) a_v\lambda_v. \tag{26}$$

Then, for $n \geq 1$, we have

$$\Delta T_n = T_n - T_{n-1} = \frac{p_n^{\alpha}}{P_n^{\alpha} P_{n-1}^{\alpha}} \sum_{v=1}^{n} P_{v-1} a_v\lambda_v$$

$$= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v.$$

By Abel's transformation, we have

$$\Delta T_n = \frac{n+1}{nP_n^\alpha} p_n^\alpha t_n \lambda_n - \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{v=1}^{n-1} p_v^\alpha t_v \lambda_v \frac{v+1}{v}$$

$$+ \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{v=1}^{n-1} P_v^\alpha t_v \Delta\lambda_v \frac{v+1}{v} + \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \sum_{v=1}^{n-1} P_v^\alpha t_v \lambda_{v+1} \frac{1}{v}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \tag{27}$$

In order to prove that the theorem works, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)} |\Delta T_n|^k < \infty. \tag{28}$$

Using Minkowski's inequality

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k),$$

Equation 28 is reduced to

$$\sum_{n=1}^{\infty} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)} |T_{n,r}|^k < \infty \; for \; r = 1,2,3,4. \tag{29}$$

Now the L.H.S of Equation 29 is given as

$$\sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)} |T_{n,1}|^k$$

$$= \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)} \left| \frac{n+1}{nP_n^\alpha} p_n^\alpha t_n \lambda_n \right|^k$$

$$= \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)-k} |t_n|^k |\lambda_n|$$

$$= O(1) |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n^\alpha}{p_n^\alpha} \right)^{\gamma(\delta k+k-1)-k} |t_n|^k$$

$$+O(1)\sum_{n=1}^{m-1}\Delta\,|\,\lambda_n\,|\sum_{v=1}^{n}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}\qquad |\,t_v\,|^k$$

$$=O(1)\,|\,\lambda_m\,|\,X_m+O(1)\sum_{n=1}^{m-1}|\,\Delta\lambda_n\,|\,X_n$$

$$=O(1)\,|\,\lambda_m\,|\,X_m+O(1)\sum_{n=1}^{m-1}\beta_n X_n$$

$$=O(1)\ \text{as}\ m\to\infty, \tag{30}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)}\qquad |\,T_{n,2}\,|^k$$

$$=\sum_{n=2}^{m+1}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n^{\alpha}}{P_n^{\alpha}P_{n-1}^{\alpha}}\sum_{v=1}^{n-1}p_v^{\alpha}t_v\lambda_v\,\frac{v+1}{v}\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}^{\alpha}}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}\qquad \sum_{v=1}^{n-1}p_v^{\alpha}\,|\,\lambda_v\,||\,t_v\,|^k\times$$

$$\times\left(\frac{1}{P_{n-1}^{\alpha}}\sum_{v=1}^{n-1}p_v^{\alpha}\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}p_v^{\alpha}\,|\,\lambda_v\,||\,t_v\,|^k\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}^{\alpha}}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}$$

$$=\sum_{v=1}^{m}p_v^{\alpha}\,|\,\lambda_v\,||\,t_v\,|^k\,\frac{1}{P_v^{\alpha}}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}$$

$$=\sum_{v=1}^{m}|\,\lambda_v\,||\,t_v\,|^k\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}$$

$$=O(1)\,|\,\lambda_m\,|\sum_{n=1}^{m}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}\qquad |\,t_n\,|^k$$

$$+O(1)\sum_{n=1}^{m-1}\Delta\,|\,\lambda_n\,|\sum_{v=1}^{n}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}\qquad |\,t_v\,|^k$$

$$=O(1)\,|\,\lambda_m\,|\,X_m+O(1)\sum_{n=1}^{m-1}|\,\Delta\lambda_n\,|\,X_n$$

$$=O(1)\,|\,\lambda_m\,|\,X_m+O(1)\sum_{n=1}^{m-1}\beta_n X_n$$

$$=O(1)\ \text{as}\ m\to\infty, \tag{31}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)}\qquad |\,T_{n,3}\,|^k$$

$$=\sum_{n=2}^{m+1}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n^{\alpha}}{P_n^{\alpha}P_{n-1}^{\alpha}}\sum_{v=1}^{n-1}P_v^{\alpha}t_v\Delta\lambda_v\,\frac{v+1}{v}\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}^{\alpha}}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}\qquad \sum_{v=1}^{n-1}P_v^{\alpha}\beta_v\,|\,t_v\,|^k\times$$

$$\times\left(\frac{1}{P_{n-1}^{\alpha}}\sum_{v=1}^{n-1}P_v^{\alpha}\beta_v\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{m}P_v^{\alpha}\beta_v\,|\,t_v\,|^k\ \sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}^{\alpha}}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)-k}$$

$$=O(1)\sum_{v=1}^{m}P_v^{\alpha}\beta_v\,|\,t_v\,|^k\ \frac{1}{P_v^{\alpha}}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}$$

$$=O(1)\sum_{v=1}^{m}v\beta_v\,\frac{1}{v}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}\qquad |\,t_v\,|^k$$

$$=m\beta_m\sum_{v=1}^{m}\frac{1}{v}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}\qquad |\,t_v\,|^k$$

$$+O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\frac{1}{i}\left(\frac{P_i^\alpha}{p_i^\alpha}\right)^{1-k+\gamma(\delta k+k-1)} \quad |t_i|^k$$

$$= O(1)m\beta_m X_m + O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v$$

$$= O(1)m\beta_m X_m + O(1)\sum_{v=1}^{m-1}X_v|\Delta\beta_v|$$

$$+O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}$$

$$= O(1) \text{ as } m\to\infty, \tag{32}$$

$$\sum_{n=1}^{m}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\gamma(\delta k+k-1)} |T_{n,4}|^k$$

$$= \sum_{n=1}^{m}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha}\sum_{v=1}^{n-1}P_v^\alpha t_v\lambda_{v+1}\frac{1}{v}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}^\alpha}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\gamma(\delta k+k-1)-k}\sum_{v=1}^{n-1}P_v^\alpha\frac{|\lambda_{v+1}|}{v}|t_v|^k\times$$

$$\times\left(\frac{1}{P_{n-1}^\alpha}\sum_{v=1}^{n-1}P_v^\alpha\frac{|\lambda_{v+1}|}{v}\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}P_v^\alpha\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}^\alpha}\left(\frac{P_n^\alpha}{p_n^\alpha}\right)^{\gamma(\delta k+k-1)-k}$$

$$= O(1)\sum_{v=1}^{m}P_v^\alpha\frac{|\lambda_{v+1}|}{v}|t_v|^k\frac{1}{P_v^\alpha}\left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{1-k+\gamma(\delta k+k-1)}$$

$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{1}{v}\left(\frac{P_v^\alpha}{p_v^\alpha}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k$$

$$= O(1)\,|\lambda_{m+1}|\sum_{v=1}^{m}\frac{1}{v}\left(\frac{P_v^{\alpha}}{p_v^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k$$

$$+O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{i=1}^{v}\frac{1}{i}\left(\frac{P_i^{\alpha}}{p_i^{\alpha}}\right)^{1-k+\gamma(\delta k+k-1)}|t_i|^k$$

$$= O(1)\,|\lambda_{m+1}|\,X_m + O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\,X_v$$

$$= O(1)\,|\lambda_{m+1}|\,X_m + \sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}$$

$$= O(1)\ \text{as}\ m\to\infty, \tag{33}$$

In merging Equations 26 to 33, we have

$$\sum_{n=1}^{\infty}\left(\frac{P_n^{\alpha}}{p_n^{\alpha}}\right)^{\gamma(\delta k+k-1)}|T_{n,r}|^k<\infty\ \text{for}\ r=1,2,3,4. \tag{34}$$

Hence, the theorem is proven.

COROLLARIES

Corollary 5.1

Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) be such that the conditions of Equations 15 to 18 and 24 are satisfied. If the following conditions also satisfy

$$\sum_{n=v+1}^{\infty}\frac{p_n^{\alpha}}{P_n^{\alpha}P_{n-1}^{\alpha}}=O(\frac{1}{P_v^{\alpha}}), \tag{35}$$

$$\sum_{n=1}^{m}\frac{p_n^{\alpha}}{P_n^{\alpha}}|t_n|^k=O(X_m)\,\text{as}\,m\to\infty, \tag{36}$$

$$\sum_{n=1}^{m}\frac{1}{n}|t_n|^k=O(X_m)\,\text{as}\,m\to\infty, \tag{37}$$

then, the series $\sum a_n\lambda_n$ is $|\bar{N},p_n^{\alpha}|_k$ summable for $k\geq 1$.

The proof for the theorem is established thus: $\gamma = 1$ and $\delta = 0$ are used in Theorem 3.1 to get Equations 35, 36 and 37. We omitted the details as the proof was similar to that of Theorem 3.1, but we used Equations 35, 36 and 37 instead of Equations 22, 23 and 25.

Corollary 5.2

Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) be such that the conditions of Equations 15 to 18 and 24 are satisfied. If the following conditions also satisfy

$$\sum_{n=v+1}^{\infty} \frac{p_n^{\alpha}}{P_n^{\alpha} P_{n-1}^{\alpha}} = O\left(\frac{1}{P_v^{\alpha}}\right), \tag{38}$$

$$\sum_{n=1}^{m} \frac{p_n^{\alpha}}{P_n^{\alpha}} |t_n| = O(X_m) \text{ as } m \to \infty, \tag{39}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n| = O(X_m) \text{ as } m \to \infty, \tag{40}$$

then, the series $\sum a_n \lambda_n$ is $|\overline{N}, p_n^{\alpha}|$ summable.

The proof for this theorem is obtained thus: by replacing $\gamma = 1$, $\delta = 0$ and $k = 1$ in Theorem 3.1, we got Equations 38, 39 and 40. We omitted the details as the proof was similar to that of Theorem 3.1, but we used Equations 38, 39 and 40 instead of Equations 22, 23 and 25.

CONCLUSION

A necessary and sufficient condition for a system to be BIBO (Bounded Input Bounded Output) stable is that the impulse response must be absolutely summable i.e.

$$\text{BIBOstable} \Leftrightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Thus, the summability technique was used in this study to stabilise and bind the output of signals and to predict the behaviour of input data, the initial situation and the changes in the completed process. This study also has a number of direct applications in the rectification of signals in the FIR filter (finite impulse response filter) and the IIR filter (infinite impulse response filter). The results of this investigation are generalised and can be reduced for several well-known summabilities.

ACKNOWLEDGEMENT

The authors express their sincere gratitude to the Department of Science and Technology, India for providing financial support to the second author under the INSPIRE Scheme (Innovation in Science Pursuit for Inspired Research Scheme).

REFERENCES

Bor, H. (2014). A new theorem on the absolute Riesz summability factors. *Filomat*, *28*(8), 1537–1541.

Bor, H. (2015). Some new results on infinite series and Fourier series. *Positivity*, *19*(3), 467–473.

Bor, H., & Özarslan, H. S. (2002). On the quasi power increasing sequences. *Journal of Mathematical Analysis and Applications*, *276*(2), 924–929.

Bor, H., & Seyhan, H. (1999). On almost increasing sequences and its applications. *Indian Journal Pure Applied Mathematics, 30,* 1041–1046.

Flett, T. M. (1957). On an extension of absolute summability and some theorems of Littlewood and Paley. *Proceedings of the London Mathematical Society, 3*(1), 113–141.

Mazhar, S. M. (1997). A note on absolute summability factors. *Bulletin-Institute of Mathematics Academia Sinica*, *25*, 233–242.

Özgen, H. N. (2013). A note on generalized absolute summability. *Annals of the Alexandru Ioan CUZA University - Mathematics*, *59*(1), 185–190.

Sonker, S., & Munjal, A. (2016a). Absolute summability factor $\varphi-|C,1,\delta|_k$ of infinite series. *International Journal of Mathematical Analysis, 10*(23), 1129–1136.

Sonker, S., & Munjal, A. (2016b). Sufficient conditions for triple matrices to be bounded. *Nonlinear Studies, 23*(4), 533–542.

Sonker, S., & Munjal, A. (2017a). A note on boundness conditions of absolute summability $\varphi-|A|_k$ factors. *Proceedings of the International Conference on Advancements in Science and Technology-2017, 67,* 208–210.

Sonker, S., & Munjal, A. (2017b). Absolute summability $\varphi-|C,\alpha,\beta;\delta|_k$ of infinite series. *Journal of Inequalities and Applications, 168,* 1–7.

Sonker, S., & Munjal, A. (2017c). Absolute summability factor $|N,p_n|_k$ of improper integrals. *International Journal of Engineering and Technology, 9*(3S), 457–462.

Sonker, S., & Munjal, A. (2017d). Absolute Nörlund summability $|N,p_n|_k$ of improper integrals. *NCRAME-2017, 2*(90), 413–415.

Sonker, S., Krasniqi, X. Z., & Munjal, A. (2017). A note on absolute Cesáro $\varphi-|C,1;\delta;l|_k$ summability factor. *International Journal of Analysis and Applications, 15*(1), 108–113.